Appendix C: Solving for Asset Prices Using Projection Methods

I formulate the PDEs for the N-state case, even though the empirical section of this paper calibrates to a 2-state model. The beliefs of type 2 agents as a function of type 1's innovation follow $d\pi_t^{(2)} = [\,\mu_t^{(2)} - \sigma_t^{(2)}\sigma_{\eta t}\,]\;dt \;+\; \sigma_t^{(2)} d\tilde{W}_t^{(1)}$. Now using (3) and substituting the definition of the pricing kernel in (9), into (27), and using Ito's Lemma, leads to the PDE

$$
\begin{aligned}
0 \;&=\; P_{\pi^{(1)}}\left(\mu^{(1)} - \sigma^{(1)}\phi^{(1)^{\top}}\right) \;+\; P_{\pi^{(2)}}\left(\mu^{(2)} - \sigma^{(2)}(\sigma_\eta + \phi^{(1)^{\top}})\right) - P_\eta \eta \phi^{(1)}\sigma^\eta + P_q q(\bar{\theta}^{(1)} - \phi^{(1)}\sigma_q^\top) \\
&+\; q - r(\pi^{(1)}, \pi^{(2)}, \eta)P \;+\; \frac{1}{2}P_{\pi^{(1)}\pi^{(1)}}\sigma^{(1)}\sigma^{(1)^{\top}} \;+\; P_{\pi^{(1)}\pi^{(2)}}\sigma^{(1)}\sigma^{(2)^{\top}} \;+\; P_{\pi^{(1)}\eta}\eta\sigma^{(1)}\sigma_\eta \\
&+\; qP_{\pi^{(1)}q}\sigma^{(1)}\sigma_q^\top + \frac{1}{2}P_{\pi^{(2)}\pi^{(2)}}\sigma^{(2)}\sigma^{(2)^{\top}} \;+\; P_{\pi^{(2)}\eta}\eta\sigma^{(2)}\sigma_\eta + qP_{\pi^{(2)}q}\sigma^{(2)}\sigma_q^\top + \frac{1}{2}P_{\eta\eta}\eta^2\sigma^{\eta^{\top}}\sigma^\eta \\
&+\; qP_{\eta q}\eta\sigma_q\sigma_\eta + \frac{1}{2}q^2 P_{qq}\sigma_q\sigma_q^\top. \qquad \text{(C1)}
\end{aligned}
$$

The time derivative is omitted since the stock price is an asset with unbounded maturity. Now guessing that the stock price is homogeneous of degree one in dividends, I write $P(q, \pi^{(1)}, \pi^{(2)}, \eta) = p(\pi^{(1)}, \pi^{(2)}, \eta)q$. Substituting into (C1) I find that all terms involving q cancel, and the price-dividend ratio, $p(\cdot,\cdot)$, must satisfy

$$
\begin{aligned}
0 \;&=\; p_{\pi^{(1)}}\left(\mu^{(1)} + \sigma^{(1)}\,(\sigma_q - \phi^{(1)^{\top}})\right) \;+\; p_{\pi^{(2)}}\;\left(\mu^{(2)} + \sigma^{(2)}(\sigma_q^\top - \phi^{(1)^{\top}} - \sigma_\eta)\right) + p_\eta\eta\;\left((\sigma_q - \phi^{(1)})\sigma_\eta\right) \\
&+\; p\,(\bar{\theta}^{(1)} - \phi^{(1)}\sigma_q^\top) + 1 - r(\pi^{(1)}, \pi^{(2)}, \eta)\,p \;+\; \frac{1}{2}p_{\pi^{(1)}\pi^{(1)}}\sigma^{(1)}\sigma^{(1)^{\top}} \;+\; p_{\pi^{(1)}\pi^{(2)}}\sigma^{(1)}\sigma^{(2)^{\top}} \\
&+\; \frac{1}{2}p_{\pi^{(2)}\pi^{(2)}}\sigma^{(2)}\sigma^{(2)^{\top}} \;+\; p_{\pi^{(1)}\eta}\eta\sigma^{(1)}\sigma_\eta \;+\; p_{\pi^{(2)}\eta}\eta\sigma^{(2)}\sigma_\eta + \frac{1}{2}p_{\eta\eta}\eta^2\sigma^{\eta^{\top}}\sigma^\eta, \qquad \text{(C2)}
\end{aligned}
$$

where the partial derivatives with respect to beliefs of type m, are written in vector form, for example, $p_{\pi^{(m)}} = \left(p_{\pi_1^{(m)}}, p_{\pi_2^{(m)}}, \cdots, p_{\pi_N^{(m)}}\right)$. In this paper I solve only for prices in the two-state case, and hence consider only the partial derivatives with respect to $\pi_1^{(m)}$. The bond price-coupon ratio satisfies a similar PDE (eq. (C3)). Boundary conditions for these PDEs are discussed next.

Proposition 4 *The following pricing multiples obtain in economies inhabited solely by agents of type m:*

(i) The equity price-dividend ratio:

$$p^{(m)}(\pi_t^{(m)}) = \sum_{i=1}^{N} C_i^{(m)} \cdot \pi_{it}^{(m)},$$

in which $C_i^{(m)}$ are positive constants satisfying $\hat{\theta}_i^{(m)} \cdot C_i^{(m)} = 1 + \sum_{j=1}^{N} \lambda_{ij}^{(m)} \cdot C_j^{(m)}$, and $\hat{\theta}_i^{(m)} = \rho + (1-\gamma)\kappa_i^{(m)} - \theta_i^{(m)} + (1-\gamma)\sigma_q\sigma_x^\top - \frac{1}{2}(1-\gamma)(2-\gamma)\sigma_x\sigma_x^\top$.

(ii) The bond price-coupon ratio:

$$b^{(m)}(\pi_t^{(m)}) = \sum_{i=1}^{N} D_i^{(m)} \cdot \pi_{it}^{(m)},$$

in which $D_i^{(m)}$ are positive constants satisfying $\hat{r}_i^{(m)} \cdot D_i^{(m)} = 1 + \sum_{j=1}^{N} \lambda_{ij}^{(m)} \cdot D_j^{(m)}$, and $\hat{r}_i^{(m)} = \rho + (1-\gamma)\kappa_i^{(m)} - \frac{1}{2}(1-\gamma)(2-\gamma)\sigma_x\sigma_x^\top$.

Proof. Follows from straightforward extensions of Proposition 1 in David and Veronesi (2002). See also Veronesi (2000).

In the homogeneous agent benchmark economies, pricing multiples are the conditionally expected values of these same multiples in each of the N possible states. For example,

$$C_i^{(m)} = \frac{1}{u_c^{(m)}(c_t^{(m)})q_t} E_t^{(m)}[\int_t^\infty u_c^{(m)}(c_s^{(m)})q_s ds | \nu_t = \nu_i^{(m)}],$$

is type m agents' expectation of future dividends conditional on the state at t being $\nu_i^{(m)}$, discounted by the marginal utility of consumption. I note that in the homogeneous agent economies, $u_c^{(m)}(c_t^{(m)}) = \xi_t^{(m)}$, where $\xi_t^{(m)}$ is still specified as in (9), but where the interest rate simplifies as in (20) with the last term set to zero, and the market prices of risk collapse to $\phi^{(m)} = (1-\gamma)\sigma_x$.

I now specify the conditions at the boundaries of the disagreement value and the belief process for the PDE (C2). $p(\pi^{(1)}, \pi^{(2)}, 0) = p^{(1)}(\pi^{(1)})$, and $\lim_{\eta \to \infty} p(\pi^{(1)}, \pi^{(2)}, \eta) = p^{(2)}(\pi^{(2)})$. The latter condition is imposed at an arbitrarily chosen large value for η. The results in the paper use an upper bound for η of 25, while empirically, I estimate the $\{\eta_t\}$ process to have been in the interval $(0, 2.5)$ in my sample from 1971-2001. David (1997) shows that 0 and 1 are entrance boundaries for each of the belief processes $\pi^{(m)}$. That is 0 and 1 cannot be attained from the interior of the belief state space. For such state variables, boundary conditions at 0 and 1 are endogenously generated by substituting $\pi^{(m)} = 0(1)$, into (C2).

Following similar steps I can show that consol price can be written as $B(\pi^{(1)}, \pi^{(2)}, \eta) = b(\pi^{(1)}, \pi^{(2)}, \eta) \cdot c$, and $b(\cdots)$ satisfies the PDE

$$
\begin{aligned}
0 &= b_{\pi^{(1)}}(\mu^{(1)} - \sigma^{(1)}\phi^{(1)\top}) + b_{\pi^{(2)}}\left(\mu^{(2)} - \sigma^{(2)}(\sigma_\eta + \phi^{(1)\top})\right) - b_\eta \eta \sigma^\eta \phi^{(1)\top} \\
&+ 1 - r + \frac{1}{2} b_{\pi^{(1)}\pi^{(1)}} \sigma^{(1)} \sigma^{(1)\top} + b_{\pi^{(1)}\pi^{(2)}} \sigma^{(1)} \sigma^{(2)\top} + \frac{1}{2} b_{\pi^{(2)}\pi^{(2)}} \sigma^{(2)} \sigma^{(2)\top} \\
&+ b_{\pi^{(1)}\eta} \eta \sigma^{(1)} \sigma^\eta + b_{\eta\pi^{(2)}} \eta \sigma^{(2)} \sigma^\eta + \frac{1}{2} b_{\eta\eta} \eta^2 \sigma^{\eta\top} \sigma^\eta.
\end{aligned} \tag{C3}
$$

I use projection methods described in Judd (1992) and Judd (1999) to solve the partial differential equations (PDEs) for the consol bond to coupon ratio, the price-dividend ratio, and the ratio of wealth to aggregate output in eqs. (C3), (C2), and, (C5) respectively. I focus my discussion on (C3) since each is a 3-dimensional parabolic PDE with the same state variables $\pi_1^{(1)}$, $\pi_2^{(2)}$, and η_t.

I proceed by formulating an 'approximate' solution to (C3) using projection methods (Judd 1999, Chapter 11).

STEP 1. Choice of individual basis functions. I choose the Chebyshev polynomials in each of the 3 dimensions: The Chebysev polynomials on $[-1, 1]$ for the basis for each dimension are given by

$$q_m(x) = \cos(m \cos^{-1} x),$$

for $m = 1, 2, \cdots$, which satisfy the recursive scheme

$$q_{m+1}(x) = 2\,x q_m(x) - q_{m-1}(x). \tag{C4}$$

These polynomials are restricted for the interval $[a, b]$ using the transformation

$$p_m,(x) = \frac{q_m(\frac{2x-a-b}{b-a})}{||q_m(\frac{2x-a-b}{b-a})||}.$$

For the belief variables, $a = 0$ and $b = 1$. For η, I use the interval $[0, 25]$. The family $\{p_m(x)\}_{m=1,2,\cdots}\}$ are orthonormal polynomials over the chosen intervals.

STEP 2. Choose a basis of 'complete' polynomials over the space $[0, 1]^2 \times [0, 25]$. The basis of degree M over the 3 dimensions is given by

$$\mathcal{P}_M = \{p_{1,i_1}(\pi_1^{(1)}) \cdot p_{2,i_2}(\pi_1^{(2)}) \cdot p_{3,i_3}(\eta) | \sum_{n=1}^{3} i_n \;\leq\; M, 0 \leq i_1, \cdots, i_3\}$$

I will write the generic element of $\mathcal{P}_M^N$ as $\phi_m(\pi_1^{(1)}, \pi_1^{(2)}, \eta)$, $m = 1, 2, \cdots M^c$, where M^c is the length of the complete polynomial basis. The set of complete polynomials for an N dimensional problem grows polynomially in N, as opposed to the tensor product basis which would use every possible product of the degree-M individual basis functions, and hence would grow at the rate of M^N (see, e.g., pp. 239 in Judd 1999). The complete polynomials asymptotically, as M becomes large, provide as good an approximation as the tensor product, but with far fewer elements. For example, I solve each PDE using $M = 15$. Using the tensor product basis, I would have a total of 3375 elements, but using the complete basis, I have far fewer, 816 elements. Extending the L^2 norm over the 3-dimensional space as the 3-fold integral, it can be verified that the basis of complete polynomials is orthonormal on $[0, 1]^2 \times [0, 25]$.

STEP 3 Let $\mathcal{D}(y)$ be the differential operator associated with the PDE (C3), i.e.

$$\begin{aligned}\mathcal{D}(y) &= y_{\pi^{(1)}}(\mu^{(1)} - \sigma^{(1)}\phi^{(1)\top}) + y_{\pi^{(2)}}\left(\mu^{(2)} - \sigma^{(2)}(\sigma_\eta + \phi^{(1)\top})\right) - y_\eta \eta \sigma^\eta \phi^{(1)\top} \\ &+ 1 - r + \frac{1}{2} y_{\pi^{(1)}\pi^{(1)}} \sigma^{(1)}\sigma^{(1)\top} + y_{\pi^{(1)}\pi^{(2)}} \sigma^{(1)}\sigma^{(2)\top} + \frac{1}{2} y_{\pi^{(2)}\pi^{(2)}} \sigma^{(2)}\sigma^{(2)\top} \\ &+ y_{\pi^{(1)}\eta} \eta \sigma^{(1)}\sigma^\eta + y_{\eta\pi^{(2)}} \eta \sigma^{(2)}\sigma^\eta + \frac{1}{2} y_{\eta\eta} \eta^2 \sigma^{\eta\top}\sigma^\eta.\end{aligned}$$

Write the candidate solution as $\hat{y}(\pi_1^{(1)}, \pi_1^{(2)}, \eta) = \sum_{m=1}^{M^c} a_m \psi_m(\pi_1^{(1)}, \pi_1^{(2)}, \eta)$. Then any solution to the PDE, $\hat{y}$, will be written as $\mathcal{D}(\hat{y}) = 0$.

STEP 4 I appeal to the Chebyshev Interpolation Theorem (see, e.g. Judd 1992, Haan 1997) to find an approximate solution to the PDE. The approximation is made by evaluating the operator $\mathcal{D}(\hat{y})$ at a chosen set of points, and setting it equal to zero at each of these points. Each interpolation point therefore provides us a linear equation in the coefficients $(a_m)_{m=1}^{M^c}$. The chosen points for the 3-dimensional space are the Cartesian product of the zeros of the Chebysev polynomial of the highest degree chosen in Step 2 in each dimension. In general the $m+1$ zeros of the mth polynomial are given by

$$x_k = (-\cos\frac{2k-1}{2m}\pi) + 1)\frac{b-a}{2} + a, \qquad \text{for} \quad k = 0, 1, \cdots m.$$

For example by choosing polynomials of order 15 in each dimension, I obtain 3375 interpolation points. With M^I interpolation points, I have an overidentified system of equations in M^c unknown coefficients. I note that, since each of the PDEs has a forcing term (for example, the forcing term in (C3) is $1 - r(\pi_1^{(1)}, \pi_1^{(2)}, \eta)$), the system of equations is non-homogeneous. Denote the $M^I \times 1$ vector of constants from each equation as c, and the $M^I \times M^c$ coefficient matrix as A. Analogous to regression coefficients, the best-fitting set of coefficients satisfies:

$$\hat{a} = (A^\top A)^{-1} A^\top c.$$

Using the dynamics of output, beliefs and disagreement value, the PDE that must be followed by $f(\cdot)$ is:

$$\begin{aligned} 0 &= \frac{1}{1+k\,\eta^{\frac{1}{1-\gamma}}} + f^{(1)}\bar{\kappa}^{(1)} + f^{(1)}_{\pi^{(1)}}(\mu^{(1)} + \sigma^{(1)}\sigma_x^\top) + f^{(1)}_{\pi^{(2)}}(\mu^{(2)} + \sigma^{(2)}(\sigma_x^\top - \sigma_\eta)) + f^{(1)}_{\eta}\eta\sigma_x\sigma_\eta \\ &+ \frac{1}{2} f^{(1)}_{\pi^{(1)}\pi^{(1)}}\sigma^{(1)}\sigma^{(1)\top} + f^{(1)}_{\pi^{(1)}\pi^{(2)}}\sigma^{(1)}\sigma^{(2)\top} + f^{(1)}_{\pi^{(1)}\eta}\eta\sigma^{(1)}\sigma_\eta \\ &+ \frac{1}{2} f^{(1)}_{\pi^{(2)}\pi^{(2)}}\sigma^{(2)}\sigma^{(2)\top} + f^{(1)}_{\pi^{(2)}\eta}\eta\sigma^{(2)}\sigma_\eta + \frac{1}{2} f^{(1)}_{\eta\eta}\eta^2\sigma^{\eta\top}\sigma^{\eta}. \end{aligned} \tag{C5}$$

Boundary conditions for the PDE are once again specified using the valuations of homogeneous agents. As in Proposition 4, it is easily shown that the ratio of wealth of agents of type m to aggregate consumption is given by

$$f^{(m)}(\pi_t^{(m)}) = \sum_{i=1}^{N} E_i^{(m)} \cdot \pi_{it}^{(m)},$$

in which $E_i^{(m)}$ are positive constants satisfying $\hat{\kappa}_i^{(m)} \cdot E_i^{(m)} = 1 + \sum_{j=1}^{N} \lambda_{ij}^{(m)} \cdot E_j^{(m)}$, and $\hat{\kappa}_i^{(m)} = \rho - \gamma\kappa_i^{(m)} + \frac{1}{2}\gamma(1-\gamma)\sigma_x\sigma_x^\top$. Intuitively, wealth of agent of type m is formulated as the value of a security paying the entire output flow rate of the economy as a dividend. When solving the PDE for the wealth of agent 1, I use $f^{(1)}(\pi^{(1)}, \pi^{(2)}, 0) = f^{(1)}(\pi^{(1)})$, and $\lim_{\eta\to\infty} f^{(1)}(\pi^{(1)}, \pi^{(2)}, \eta) = 0$. The latter condition results because agents 1's consumption flow tends to zero as η becomes large. Once again it is implemented by imposing the value zero at an arbitrarily chosen large upper bound for η.

Asset Volatilities and Portfolio Choices

Since these solutions are functions of smooth polynomials, I can also use their derivatives to provide approximated solutions for volatilities, risk premia, and, portfolio choices. The volatilities of stock and bond returns with respect to the two shocks are

$$\sigma_P = \frac{p_{\pi^{(1)}}\sigma^{(1)} + p_{\pi^{(2)}}\sigma^{(2)} + p_\eta \eta \sigma_\eta}{p} + \sigma_q; \qquad \sigma_B = \frac{b_{\pi^{(1)}}\sigma^{(1)} + b_{\pi^{(2)}}\sigma^{(2)} + b_\eta \eta \sigma_\eta}{b}. \quad \text{(C6)}$$

It is useful to note that only the last term of the stock volatility expression arises from the volatility of dividends, the fundamental volatility. The first two terms arise from the volatility of the belief processes of the two types of agents, while the third term is the volatility from the disagreement value process. Since coupons are fixed, the bond price has similar terms but no volatility from fundamentals.

The volatilities of type 1's wealth with respect to the two shocks can be written as

$$\sigma_X^{(1)} = \frac{f^{(1)}_{\pi^{(1)}}\sigma^{(1)} + f^{(1)}_{\pi^{(2)}}\sigma^{(2)} + f^{(1)}_\eta \eta \sigma_\eta^\top}{f^{(1)}} + \sigma_x. \quad \text{(C7)}$$

Let the dollar positions of investor 1 in stocks and bonds be given by $w_t^{(1)} \equiv (w_{B_t}^{(1)}, w_{P_t}^{(1)})$. With these portfolio choices, the Ito representation of $X_t^{(1)}$ is

$$dX_t^{(1)} = X_t^{(1)} r_t dt + w_t^{(1)} (\mu_{B_t} - r_t, \mu_{P_t} - r_t)^\top dt + w_t^{(1)} (\sigma_{B_t}^\top, \sigma_{P_t}^\top)^\top d\tilde{W}_t^{(1)} - c_t^{(1)} dt, \quad \text{(C8)}$$

where the volatilities σ_{B_t} and σ_{P_t} are in (C6). Since the volatilities of wealth in (C7) and (C8) must be the same, the portfolio choices must satisfy (29).

Appendix D: GMM Estimation of the Regime Switching Model

In this Appendix, I provide the details of the GMM estimation that uses information in both fundamentals and a series of forecasters' dispersion about future earnings from surveys to estimate the parameter values and time-series of investors beliefs about the hidden states.

The moments that I fit jointly in my GMM procedure are the scores of the likelihood function for each agent (similar to maximum likelihood), and the dispersion process. I then find the two best sets of parameters that jointly maximize the likelihood of each type of agent observing the fundamental processes, as well as the dispersion among them. The model based dispersion is generated by formulating beliefs for each agent for each quarter, once again by Bayes' law, and taking the standard deviation cross-sectionally, as is done for the data series in eq. (31). I also use a discretized version of (25) to find the calibrated disagreement value process. I note, that because three volatility parameters are common to each agent, the number of moments exceeds the number of fitted parameters by four, implying that the GMM objective has a chi-squared distribution with four degrees of freedom. The optimized value serves as a specification test of the model.

First, let $\Psi^{(m)}$ be the set of parameters characterizing the fundamental processes (1) and (2) for each type of investor. The discretized versions of these processes are

$$q_{t+1} = q_t \cdot e^{(\theta_t^{(m)} - \frac{1}{2}\sigma_q \sigma_q')\Delta t + \sigma_q W_{t+1}} \; ; x_{t+1} = x_t \cdot e^{(\kappa_t^{(m)} - \frac{1}{2}\sigma_x \sigma_x')\Delta t + \sigma_x W_{t+1}}.$$

Let $\mathcal{G}^{(m)}(T)$ be the filtration on the set of unobserved drift states generated by the time series of realized fundamentals for investors of type m, $m = 1, 2$. I note that the filtrations on the underlying state spaces will differ even though each agent observes the same set of fundamentals, due to the differences in their parameter estimates $\nu^{(m)}$ and $\Lambda^{(m)}$. I assume that the volatility parameters in parameters of the fundamental processes, Σ, are common to the two agents.

The updating process is discretized as follows: Let $\pi^{(m)}(t|t) = \left(\pi_1^{(m)}(t|t), ..., \pi_N^{(m)}(t|t)\right)$ be the row vector of probabilities at time t for agents of type m, after observing fundamentals at t.

Let $P^{(m)}(\Delta t) = \exp(\Lambda \cdot \Delta t)$ be the transition matrix over a non-infinitesimal interval between observations, Δt.. In my estimation technique, I estimate $P^{(m)}(0.25)$, the quarterly transition matrix. The implied generator is $\Lambda^{(m)} = \sum_{i=1}^{\infty}(-1)^{i+1} \cdot \left((P^{(m)}(0.25))^4 - I_2\right)^i / i$ (see Israel, Rosenthal, and Wei 2001), whose value I estimate using a series approximation of length 10. A straightforward application of Bayes law implies that the updating rule for the posterior distribution on the state space $\nu^{(m)} = \left(\nu_1^{(m)}, ..., \nu_N^{(m)}\right)$ when the time between observations is Δt (see Hamilton 1989, Hamilton 1994):

$$\pi_i^{(m)}(t|t) = \frac{e^{-\frac{1}{2}\left(\Delta \log(y)(t) - \widehat{\nu}_i^{(m)}\Delta t\right)^\top \left(\Sigma\Sigma^\top\right)^{-1}\left(\Delta \log(y)(t) - \widehat{\nu}_i^{(m)}\Delta t\right)} \left[\pi^{(m)}(t|t-\Delta t) P^{(m)}(\Delta t)\right]_i}{\sum_{j=1}^{N} e^{-\frac{1}{2}\left(\Delta \log(y)(t) - \widehat{\nu}_j^{(m)}\Delta t\right)^\top \left(\Sigma\Sigma^\top\right)^{-1}\left(\Delta \log(y)(t) - \widehat{\nu}_j^{(m)}\Delta t\right)} \left[\pi^{(m)}(t|t-\Delta t) P^{(m)}(\Delta t)\right]_j}, \tag{D1}$$

where $y(t) = (q(t), x(t))$, $\Sigma = \left(\sigma_q^\top, \sigma_x^\top\right)^\top$, and, $\widehat{\nu}_i^{(m)} = \nu_i^{(m)} - \frac{1}{2}\left(\sigma_x\sigma_x^\top, \sigma_q\sigma_q^\top\right)$. The beliefs over the next interval with no new information are expected to be:

$$\pi_i^{(m)}(t+\Delta|t) = \pi_i^{(m)}(t|t) \cdot P^{(m)}(\Delta t). \tag{D2}$$

David (1993) shows that as the length of time between observations, Δt, goes to zero, the discrete-time process $\{\pi_i^{(m)}(t|t)\}$ converges almost surely to the diffusion process in (3).

The likelihood function of agent m over fundamentals is then given by:

$$\mathfrak{L}^{(m)}\left(\Psi^{(m)}|\mathcal{G}^{(m)}(T)\right) = \sum_{t=2}^{T} \log f^{(m)}\left(\Delta \log(y)(t)|\mathcal{G}^{(m)}(t); \Psi^{(m)}\right), \tag{D3}$$

where, $f^{(m)}\left(\Delta \log(y)(t)|\mathcal{G}^{(m)}(t); \Psi^{(m)}\right) =$

$$\sum_{i=1}^{N}\left[\pi^{(m)}(t|t-\Delta t) P^{(m)}(\Delta t)\right]_i \times e^{-\frac{1}{2}\left(\Delta \log(y)(t) - \hat{\nu}_i^{(m)}\Delta t\right)^\top \left(\Sigma\Sigma^\top \Delta t\right)^{-1}\left(\Delta \log(y)(t) - \widehat{\nu}_i^{(m)}\Delta t\right)}.$$

The priors $\pi^{(m)}(0|0)$ are taken to be the unconditional means of the states implicit in the matrix $\Lambda^{(m)}$.

As a first step, I test for the number of drift states for the two fundamental series in a standard regime-switching framework. To do this, I use likelihood ratio tests that adjust for the presence of nuisance parameters unidentified under the null hypothesis (for example, under the 1-state null hypothesis, the transition probabilities of a 2-state model are not identified). Exact critical values for the alternative of two states over the null of a single state are available in Table 1A of Garica (1998). For earnings growth, the log likelihood ratio (LLR) attains a value of 14.1622, which has a P-value of less than 1 %. The LLR for a 3-state over a 2-state specification is very small, of the order of 10^{-3}, suggesting insignificant gains in modeling a third state for the earnings drift. For aggregate consumption growth, the LLR for a 2-state over a 1-state model attains a value of 5.9212, with a P-value that lies between 50 and 70 %. This is in line with the findings of other authors that aggregate consumption growth is nearly unpredictable (see Hall 1978). Therefore, I fit a model with two drift states of earnings and one for aggregate consumption growth.

To obtain heterogeneous parameters of the fundamental processes, we obtain information from the time series of dispersion of earnings growth. Given the beliefs of agents of each type at time t, $\pi^{(m)}(t|t)$, we write the beliefs over earnings growth τ periods ahead (each period of length Δt) as:

$$\pi^{(m)}(t,\tau) = \pi^{(m)}(t|t) \cdot (P^{(m)}(\Delta t))^{\tau}. \tag{D4}$$

Then I form the expected growth rate of earnings τ periods ahead for each agent, and the standard deviation of the cross-sectional earnings expectations to obtain a series of model-generated dispersion. Call $d(t,\tau)$, the τ quarters ahead dispersion in the data from eq. (31), and its model-generated counterpart $\hat{d}(\pi^{(1)}(t,\tau), \pi^{(2)}(t,\tau))$. I define the dispersion error as:

$$e\,(t,\tau) = [d(t,\tau) - \hat{d}(\pi^{(1)}(t,\tau), \pi^{(2)}(t,\tau)].$$

I simultaneously estimate the two sets of parameters $\Psi^{(m)}$, $m = 1, 2$ from a GMM procedure with the following errors: $\epsilon(t)^\top = \{e(t,\tau), \frac{\partial \mathfrak{L}^{(1)}}{\Psi^{(1)}}^\top(t), \frac{\partial \mathfrak{L}^{(2)}}{\Psi^{(2)}}^\top(t)\}$, where the second and third terms are the scores of the likelihood function of fundamentals with respect to $\Psi^{(1)}$ and $\Psi^{(2)}$ respectively. In the estimation procedure, I use $\Delta t = 1/4$ and $\tau = 4$. Similar results were obtained using $\tau = 1, \cdots 4$. I now form the GMM objective:

$$c = \left(\frac{1}{T}\sum_{t=1}^{T} \cdot \epsilon_{\mathbf{t}}\right)^\top \cdot \Omega^{-1} \cdot \left(\frac{1}{T} \cdot \sum_{t=1}^{T} \epsilon_{\mathbf{t}}\right).$$

Since I find the processes of dispersion errors, $\{e_t\}$, to be serially correlated, while the scores are not, I diagonally partition the matrix Ω into two parts: Ω_Ψ, and Ω_d. Ω_d is estimated using the Newey-West correction (see Hamilton 1994, Eq 14.1.19):

$$\begin{aligned}
\hat{\Omega}_{d,T} &= \hat{\Gamma}_{0,T} + \sum_{j=1}^{J}[1 - j/(J+1)] \cdot (\hat{\Gamma}_{j,T} + \hat{\Gamma}'_{j,T}), \qquad \text{where} \\
\hat{\Gamma}_{j,T} &= \frac{1}{T-J} \cdot \sum_{t=j+1}^{T} e_t \cdot e'_t.
\end{aligned}$$

Ω_Ψ is estimated by $\frac{1}{T}\sum_{t=1}^{T}\left[\frac{\partial \mathfrak{L}}{\partial \Psi}(t)^\top \frac{\partial \mathfrak{L}}{\partial \Psi}(t)\right]$, where $\frac{\partial \mathfrak{L}}{\partial \Psi}^\top(t) \equiv \left(\frac{\partial \mathfrak{L}^{(1)}}{\partial \Psi^{(1)}}^\top(t), \frac{\partial \mathfrak{L}^{(2)}}{\partial \Psi^{(2)}}^\top(t)\right)$.

I will then look for parameters $\Psi^{(m)}$ that jointly minimize the dispersion error and the scores of the likelihood function from fundamentals. Since each of the likelihood functions depends on distinct drifts and generator elements for each agent, but three common volatility parameters, the overall GMM objective is overidentified and has a $\chi^2(4)$ distribution. The value of the objective function then serves as an omnibus specification test statistic for the model.